DRAFT SF 298

1. Report Date (dd-mm-yy)	2. Report Type	3. Dates covered (from... to)

4. Title & subtitle Control of Compressor Rotating Stall without Distributed Sensing using Bifurcation Stabilization	5a. Contract or Grant # 5b. Program Element #
6. Author(s)	5c. Project # 5d. Task # 5e. Work Unit #
7. Performing Organization Name & Address	8. Performing Organization Report #
9. Sponsoring/Monitoring Agency Name & Address	10. Monitor Acronym 11. Monitor Report #

12. Distribution/Availability Statement
Distribution approved for Public Release, Distribution Unlimited

13. Supplementary Notes TR downloaded from a WWW unrestricted URL.

14. Abstract

19970121 166

15. Subject Terms

Security Classification of			19. Limitation of Abstract	20. # of Pages	21. Responsible Person (Name and Telephone #)
16. Report Unclass	17. Abstract Unclass	18. This Page Unclass	Unlimited		

Control of Compressor Rotating Stall without Distributed Sensing using Bifurcation Stabilization

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August 6, 1996

Abstract

Control of rotating stall in axial compressors is considered. A local bifurcation stabilization theorem using the projection method for the case of an uncontrollable, unobservable critical mode is described and extended to control laws that do not vanish at the critical or bifurcation point. This result is used to derive sufficient conditions for several control laws to guarantee that the subcritical pitchfork bifurcation of an axial compressor model is made supercritical so that the rotating stall hysteresis is eliminated. Each of the control laws considers operation at a set point distinct from the critical point and depends only on annulus-averaged quantities as feedback variables to simplify sensing and signal processing requirements. The actuation considered is a bleed valve so that the control law must be positive for all possible values of the feedback variables. It is shown that positive control is stabilizing for only some of the control laws considered. In these cases, numerical examples show the transformation of the bifurcation from subcritical to supercritical and the elimination of the hysteresis region. Finally, geometric interpretations of the effects of the feedback laws in the state space are described.

1 Introduction

Rotating stall in axial compressors is a phenomenon that limits the performance of gas turbine engines. It is characterized by a cell of reduced or blocked flow rotating around the annulus of the compressor at a fraction of the rotor speed. Such behavior is self-sustaining and causes a significant loss of performance with possible damage to the compressor blades from the periodic loading of the rotating stall cell. Furthermore, recovery from such a condition can be difficult, often requiring shutting down and restarting the engine. Such behavior has traditionally been avoided by requiring that the operating point of a compressor be well below the peak pressure rise, where the compressor is most likely to enter rotating stall. In this case, rotating stall is avoided by sacrificing performance to maintain safe operation.

A seminal contribution in the understanding of rotating stall dynamics was the development of the Moore-Greitzer compressor model [8], which combines a lumped-parameter surge model with a representation of the unsteady, two-dimensional (axial and circumferential) flow in the compressor annulus. Variations in the flow around the annulus of the compressor can be characterized using this model so that rotating stall dynamics can be studied. The full Moore-Greitzer model has the form of a nonlinear partial differential equation and a nonlinear ordinary differential equation in the pressure rise coefficient and the distributed mass flow coefficient around the annulus. The partial differential equation can be approximated using a Galerkin procedure, resulting in a set of three nonlinear ordinary differential equations for the pressure rise, annulus-averaged mass flow, and first spatial Fourier magnitude of the rotating stall cell. Although the third order approximation is a great simplification of the distributed model, it captures the essential physics of the compressor behavior and qualitatively demonstrates behavior seen experimentally.

The stationary and periodic solutions of the Moore-Greitzer three-state model as a function of the critical parameters have been studied in detail [7]. The stationary solutions of the equations consist of *axisymmetric equilibria* corresponding to design flow and *rotating stall equilibria* corresponding to disturbed flow or rotating stall. When the first spatial Fourier magnitude of the rotating stall cell is used as a state, a pitchfork bifurcation appears at the peak of the compressor characteristic and both branches of equilibria emanate from this point. When the bifurcation is subcritical, stable portions of the axisymmetric and rotating stall branches coexist over some range of parameters and hysteresis results. Since the domains of attraction of equilibria along the ax-

isymmetric branch near the bifurcation point are typically small, a small disturbance can perturb the system along a trajectory away from the axisymmetric equilibrium point to a locally stable rotating stall equilibrium. Hence operation near the hysteresis must be avoided; it is this hysteresis that limits the performance of the compressor.

Several researchers have considered controllers for the three state approximation of the Moore-Greitzer model with bleed valves as actuation. In this case the critical mode, rotating stall magnitude, is uncontrollable. The most notable controller was one based on feedback of the square of the rotating stall magnitude [6]. Using results from bifurcation stabilization theory, namely, the projection method, it was shown that this controller could eliminate the hysteresis region for sufficiently large gain. Furthermore, this control law was demonstrated experimentally [2]. An alternative approach was considered in [5]. Here, only annulus-averaged quantities, rather than the rotating stall magnitude, were used as feedback. In this case, the feedback law was chosen to guarantee global asymptotic stability based on a Lyapunov function.

In [3], one parameter families of nonlinear systems with uncontrollable critical modes were considered. It was shown that for smooth nonlinear output feedback, the control laws that stabilize the bifurcated solutions emanating from the critical point depend on the observability of the critical mode. In particular, different terms in the Taylor series expansion of the control law about the critical point were shown to affect stability for the different cases. In the case of an observable critical mode, the quadratic terms are required to be nonzero for the stabilization of a subcritical pitchfork bifurcation, while for an unobservable critical mode, linear terms are required to be nonzero and quadratic terms play no role. Hence, the results of [6], where the rotating stall magnitude was used as feedback so that the critical mode was observable, depend on a feedback law that is locally quadratic at the critical point, while the results of [5] use only annulus-averaged quantities and hence only the linear terms contribute to stability since the critical mode is unobservable.

In this paper, we extend the results of [3] which considers local bifurcation control by considering control laws that do not vanish at the critical point. We find that every term of a control law naturally written in a power series about some operating or set point affects the stability of the bifurcated solution at the critical point when the operating point is different from the critical point. We then exploit this fact by defining controllers satisfying a physical restriction on the bleed valve, namely, that since bleed valve actuation requires the control input to be positive, control

laws must be positive for all values of the feedback variables. We show that positive control is stabilizing for some of the controllers considered but not for others. In each case, the controllers rely on only annulus-averaged quantities, rather than distributed quantities as in the controllers of [2, 6]. Using only annulus-averaged quantities simplifies sensing and signal processing requirements. We present numerical examples to demonstrate the elimination of the hysteresis region when the stability conditions are satisfied and give some geometric interpretations of the effects of different control laws in the three-dimensional state space.

2 Moore-Greitzer Model

The Moore-Greitzer model consists of a compressor modelled as a semi-actuator disk in a duct, a plenum representing the combustion chamber, and a throttle representing the turbine [8]. A schematic of the model is shown in Figure 1. Here p_T is the pressure ahead of the entrance, p_s is the pressure in the plenum, $\phi(\theta, \xi)$ is the local mass flow coefficient, ξ is time nondimensionalized by rotor speed U, η is the axial spatial variable, and θ is the circumferential spatial variable.

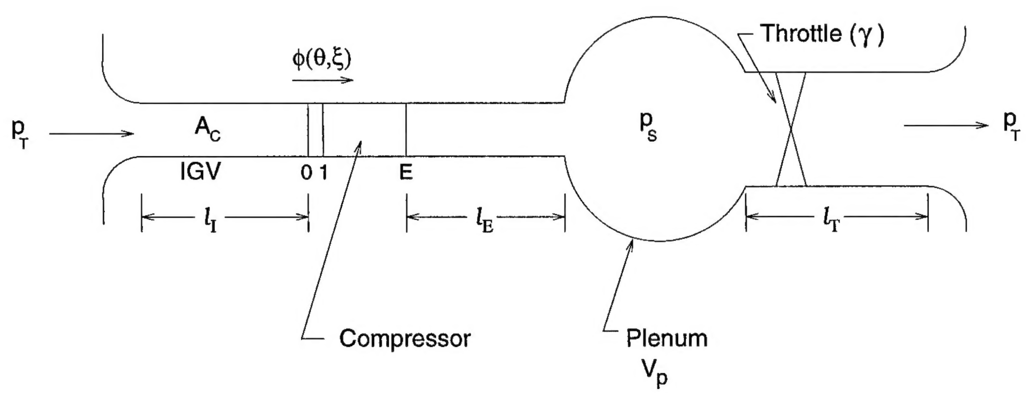

Figure 1: Schematic of Moore-Greitzer Model

Summing the pressure rise across each of the elements yields

$$\Psi + l_c \frac{d\Phi}{d\xi} = \psi_c(\phi) - \lambda \left(\frac{\partial \delta \tilde{\phi}}{\partial \xi} \right) \bigg|_{\eta=0} - \frac{1}{2a} \left(2 \frac{\partial \delta \phi}{\partial \xi} + \frac{\partial \delta \phi}{\partial \theta} \right) \bigg|_{\eta=0},$$

where $\Psi = (p_s - p_T)/\rho U^2$ is the pressure rise coefficient, Φ is the average flow coefficient around the annulus, $\delta\phi$ is the disturbance flow coefficient, $\delta\tilde{\phi}$ is the disturbance velocity potential, and $\psi_c(\Phi)$ is the compressor characteristic that relates the mass flow through the compressor to the

pressure rise in the steady state. The disturbance velocity potential satisfies Laplace's equation $\delta\tilde{\phi}_{\eta\eta} + \delta\tilde{\phi}_{\theta\theta} = 0$ so that

$$\delta\tilde{\phi} = \sum_{n=1}^{\infty} \frac{1}{n}(A_n \cos n\theta + B_n \sin n\theta)e^{n\eta},$$

so that the disturbance velocity is

$$\delta\phi = \frac{\partial\delta\tilde{\phi}}{\partial\eta} = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)e^{n\eta}.$$

Next, using conservation of mass yields the equation

$$\frac{d\Psi}{d\xi} = \frac{1}{4B^2 l_c}(\Phi - \Phi_T(\gamma, \Psi)),$$

whee $\Phi_T(\gamma, \Psi) = \gamma\sqrt{\Psi}$ is the throttle characteristic with throttle parameter γ.

By assuming a cubic compressor characteristic and taking only the first harmonic terms in the expressions for the disturbance velocity and velocity potential, a Galerkin procedure yields the third order approximation of the Moore-Greitzer model. Letting $A = \sqrt{A_1^2 + B_1^2}$, the third order model can be written as

$$\begin{aligned}
\frac{dA}{d\xi} &= \alpha A(\psi_c'(\Phi) + \frac{A^2}{8}\psi_c'''(\Phi)), && (1)\\
\frac{d\Phi}{d\xi} &= \frac{1}{l_c}(\psi_c(\Phi) - \Psi + \frac{A^2}{4}\psi_c''(\Phi)), && (2)\\
\frac{d\Psi}{d\xi} &= \frac{1}{4B^2 l_c}(\Phi - \Phi_T(\gamma, \Psi)). && (3)
\end{aligned}$$

The parameter α is related to the time lag in the blade passage, l_c is the duct length parameter, and B is the rotor speed parameter.

By setting the derivatives in (1)-(3) to zero, the equilibria of the three state model can be computed. Clearly, $A = 0$ is a possible solution to the equilibrium conditions, in this case corresponding to the design flow or axisymmetric equilibrium where

$$\begin{aligned}
\Psi &= \psi_c(\Phi),\\
\Phi &= \gamma\sqrt{\Psi}.
\end{aligned}$$

For some values of the throttle parameter γ, there also exists another equilibrium for which

$$\begin{aligned}
A^2 &= \frac{-8\psi_c'(\Phi)}{\psi_c'''(\Phi)}\\
\Psi &= \psi_c(\Phi) - \frac{2\psi_c'(\Phi)\psi_c''(\Phi)}{\psi_c'''(\Phi)},\\
\Phi &= \gamma\sqrt{\Psi},
\end{aligned}$$

which is the rotating stall equilibrium. In the sequel we will be interested in the stability of this branch of equilibria, as it will dictate the behavior of the overall system.

3 Bifurcation Stabilization

In this section we discuss stabilization of the bifurcated solution of a one-parameter family of nonlinear systems. In particular, we are interested in the case when the critical mode of the linearization is uncontrollable and unobservable. Previous results for this family of nonlinear systems are for local controllers, that is, controllers that are zero at the critical point and hence consider the critical point to be the equilibrium point. We extend these results to consider general equilibrium points away from the critical point, and show how these controllers affect the stability of the bifurcated solution emanating from the critical point.

3.1 Projection Method

First, we show a result that guarantees the stability of a one parameter family of nonlinear systems. The results presented in this section are taken from [1, 4, 6]. Consider the nonlinear system,

$$\dot{x} = f(x, \mu), \tag{4}$$

where $x \in \mathbb{R}^n$ is the state of the system, $\mu \in \mathbb{R}$ is a parameter, and $f(x_0, \mu) = 0$ so that $x = x_0$ is the equilibrium point of the system. Furthermore, the Jacobian matrix $D_x f(x_0, \mu_0)$ is singular so that $(x, \mu) = (x_0, \mu_0)$ is the critical or bifurcation point. Hence, this system may possess several equilibrium branches emanating from the critical point.

We are interested in the stability of the bifurcated equilibrium branch near the bifurcation point. Using the projection method, we can characterize the local stability of the bifurcated solution by approximating the eigenvalue of the linearization passing through the origin at the bifurcation point as

$$\lambda = \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \cdots,$$

and determine stability based on the sign of the first nonzero term. Expanding (4) in a Taylor series about the bifurcation point $x = x_0$ yields

$$\dot{x} = L_0(x - x_0) + Q_0[x - x_0, x - x_0] + C_0[x - x_0, x - x_0, x - x_0] + \cdots,$$

where linear term L_0 is

$$L_0 = \frac{\partial f}{\partial x}(x_0, \mu_0),$$

and the quadratic and cubic terms are symmetric forms

$$Q_0[x,y] = \begin{bmatrix} \vdots \\ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_0,\mu_0)x_j y_k \\ \vdots \end{bmatrix},$$

$$C_0[x,y,z] = \begin{bmatrix} \vdots \\ \frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l}(x_0,\mu_0)x_j y_k z_l \\ \vdots \end{bmatrix},$$

where x_j, y_k, and z_l are the jth, kth, and lth components of the vectors x, y, and z. It follows that L_0 has a zero eigenvalue corresponding to the critical mode. Define the left and right eigenvectors of the zero eigenvalue of L_0 as l and r, respectively, where the first component of r is chosen to be one and $lr = 1$. Then, λ_1 and λ_2 can be defined as

$$\begin{aligned} \lambda_1 &= lQ_0[r,r], \\ \lambda_2 &= 2l\left(2Q_0[r,\delta] + C_0[r,r,r]\right), \end{aligned}$$

where $l\delta = 0$ and

$$L_0\delta + Q_0[r,r] = 0.$$

The following theorem characterizes the stability of the bifurcated solutions of the system.

Theorem 3.1 *[6] The bifurcated solutions of the system (4) for μ near zero are asymptotically stable (unstable) if $\lambda_1 = 0$ and $\lambda_2 < 0$ ($\lambda_2 > 0$) so that the bifurcation is supercritical (subcritical).*

3.2 Generalization of Local Output Feedback

In [3], existence conditions are derived for a controller to stabilize the bifurcated solution of a one parameter family of nonlinear systems using local control when the critical mode is uncontrollable and unobservable. Consider the system

$$\dot{x} = f(x,\mu) + g(x)u, \;\; u = h(y), \;\; y = cx, \tag{5}$$

where the control input u is a scalar. The critical point of the system is $(x, \mu) = (x_0, \mu_0)$, and when $h(y_0) = 0$, where $y_0 = cx_0$, then this control is local, that is, the critical or bifurcation point is taken as the equilibrium point and the control vanishes there. The control law can be written in a Taylor series expansion about the critical point, and it has been shown [3, Theorem 3.2] that when the critical mode is unobservable, only the linear term of the control law $u = h(y)$ affects the stability of the bifurcated solution.

Here, we consider the more general case. Again consider the system and feedback law (5). Let x_e be the operating or set point, where $y_e = cx_e$, satisfying

$$0 = f(x_e, \mu_e) + g(x_e)h(y_e), \quad h(y_e) = 0,$$

and let x_0 be the critical or bifurcation point satisfying

$$0 = f(x_0, \mu_0) + g(x_0)h(y_0),$$

where $h(y_0)$ may be nonzero. We are interested in the case when the linearized dynamics at the critical point have a single zero eigenvalue and a bifurcation occurs there. We want to operate at the point x_e with zero control effort there and we want to stabilize the bifurcated solution emanating from the critical point x_0. Hence, we want to determine the conditions on the feedback law $u = h(y)$ to guarantee stability of the bifurcated solution. We know that for local controllers, that is when $h(y_0) = 0$, only the linear term in the Taylor series expansion about the critical point affects stability. Now we need to determine which terms affect stability if $h(y_0) \neq 0$. We will see that the constant and linear terms of the Taylor series expansion about the critical point affect the stability of the bifurcated solution so that all of the terms in the Taylor series expansion about the operating point x_e affect stability. Hence, *each* of the terms in the control law affects stability if the control law is written in terms of a set point.

Consider a control law $h(y)$ written in the form

$$h(y) = k_1(y - y_e) + k_2(y - y_e)^{(2)} + \ldots \tag{6}$$

where $z^{(p)} = [z_1^p \cdots z_n^p]^{\mathrm{T}}$ is an element-by-element exponentiation. Note that this control law is written in the form of a Taylor series expansion about the set point. Now, expand the control law in a Taylor series about the critical point

$$h(y) = \tilde{k}_0 + \tilde{k}_1(y - y_0) + \tilde{k}_2(y - y_0)^{(2)} + \cdots. \tag{7}$$

The following theorem shows that each of the terms of the control law (6) affects the stability of the bifurcated solution emanating from (x_0, μ_0) by showing that the constant and linear terms of the Taylor series expansion about the critical point determine stability. Hence, the following is a generalization of the result of [3] for local control.

Theorem 3.2 *If $x_e \neq x_0$, then every term $k_i(y - y_e)^{(i)}$ of the control law (7) affects the stability of the bifurcated solution of the system (5) emanating from (x_0, μ_0).*

Proof. Write the closed loop system as

$$\dot{x} = f(x, \mu) + g(x)[\tilde{k}_0 + \tilde{k}_1 c(x - x_0) + \tilde{k}_2 (c(x - x_0))^{(2)} + \cdots],$$

and expand in a Taylor series about the critical point

$$\dot{x} = L_0(x - x_0) + Q_0[x - x_0, x - x_0] + C_0[x - x_0, x - x_0, x - x_0] + \cdots.$$

Using the Theorem 3.1, the bifurcated solution is stable if $\lambda_2 < 0$. Here,

$$L_0 = \frac{\partial f}{\partial x}(x_0, \mu_0) + \frac{\partial g}{\partial x}(x_0)\tilde{k}_0 + g(x_0)\tilde{k}_1 c,$$

$$Q_0[x, y] = \begin{bmatrix} \vdots \\ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_0, \mu_0) + \frac{\partial^2 g_i}{\partial x_j \partial x_k}(x_0)\tilde{k}_0\right) x_j y_k \\ +\frac{1}{2}\frac{\partial g_i}{\partial x}(x_0) x \tilde{k}_1 c y + \frac{1}{2}\frac{\partial g_i}{\partial x}(x_0) y \tilde{k}_1 c x + g_i(x_0)\tilde{k}_2 (cx)\cdot(cy) \\ \vdots \end{bmatrix},$$

$$C_0[x, y, z] = \begin{bmatrix} \vdots \\ \frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l}(x_0, \mu_0) + \frac{\partial^3 g_i}{\partial x_j \partial x_k \partial x_l}(x_0)\tilde{k}_0\right) x_j y_k z_l \\ +\frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 g_i}{\partial x_j \partial x_k}(x_0)\left(x_j y_k \tilde{k}_1 c z + x_j z_k \tilde{k}_1 c y + y_j z_k \tilde{k}_1 c x\right) \\ +\frac{1}{3}\frac{\partial g_i}{\partial x}(x_0)\left(x \tilde{k}_2 (cy)\cdot(cz) + y \tilde{k}_2 (cx)\cdot(cz) + z \tilde{k}_2 (cx)\cdot(cy)\right) \\ +g_i(x_0)\tilde{k}_3 (cx)\cdot(cy)\cdot(cz) \\ \vdots \end{bmatrix},$$

where $(x)\cdot(y)$ represents element by element multiplication.

The right eigenvector corresponding to the critical mode is r, so that from observability conditions we know that if the critical mode is unobservable then $cr = 0$ so that the equations to determine λ_1 and λ_2 can be written as

$$\lambda_1 = lQ_0[r,r] = l \begin{bmatrix} \vdots \\ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_0,\mu_0) + \frac{\partial^2 g_i}{\partial x_j \partial x_k}(x_0)\tilde{k}_0\right) r_j r_k \\ \vdots \end{bmatrix}$$

$$\begin{aligned} L_0\delta &= -Q_0[r,r] \\ &= -\begin{bmatrix} \vdots \\ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_0,\mu_0) + \frac{\partial^2 g_i}{\partial x_j \partial x_k}(x_0)\tilde{k}_0\right) r_j r_k \\ \vdots \end{bmatrix} \end{aligned}$$

$$\lambda_2 = 2l \begin{bmatrix} \vdots \\ \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}(x_0,\mu_0) + \frac{\partial^2 g_i}{\partial x_j \partial x_k}(x_0)\tilde{k}_0\right) r_j \delta_k + \frac{\partial g_i}{\partial x}(x_0) r \tilde{k}_1 c\delta \\ +\frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l}(x_0,\mu_0) + \frac{\partial^3 g_i}{\partial x_j \partial x_k \partial x_l}(x_0)\tilde{k}_0\right) r_j r_k r_l \\ \vdots \end{bmatrix}$$

Hence we see that only the constant and linear terms in the Taylor series expansion about the critical point affect the stability of the bifurcated solution, since these equations depend only on $\tilde{k}_0$ and $\tilde{k}_1$. By equating the expansions (6) and (7), it follows that

$$\begin{aligned} \tilde{k}_0 &= k_1(y_0 - y_e) + k_2(y_0 - y_e)^{(2)} + k_3(y_0 - y_e)^{(3)} + \cdots, \\ \tilde{k}_1 &= k_1 + 2k_2(y_0 - y_e) + 3k_3(y_0 - y_e)^{(2)} + \cdots. \end{aligned}$$

Hence, since these terms depend on k_i for all i, it follows that each of the terms in the control law (6) affects λ_1 and λ_2 so that each affects the stability of the system. ■

The previous theorem can hence be used to find control laws having particular structures to meet other requirements. In particular, for the rotating stall problem, we will choose control laws having only quadratic terms to satisfy positivity requirements on the control. Note that if the set point is equal to the critical point, then only the linear term in the control law affects the stability of the bifurcated solution.

4 Control of Rotating Stall

In this section we characterize control laws that guarantee that the bifurcated solution is made asymptotically stable, or equivalently, that the bifurcation is rendered supercritical. Specifically, we will use bleed valves as control actuation. Since a bleed valve has the same effect on the compression system as the throttle, we replace $\Phi_T(\gamma, \Psi) = \gamma\sqrt{\Psi}$ by $\Phi_T(\gamma, \Psi) = \gamma\sqrt{\Psi} + u\sqrt{\Psi}$ where $u \geq 0$ is the bleed valve position.

4.1 Stability of the Open Loop System

Let the axisymmetric equilibrium point at the peak of the compressor characteristic be (Φ_0, Ψ_0) with corresponding throttle parameter γ_0. This point is a bifurcation point and emanating from it is the rotating stall equilibrium branch. The following gives a sufficient condition for stability of this branch for the open loop system.

Theorem 4.1 *If*

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) < 0. \tag{8}$$

then the bifurcated solution of the system (1)-(3) at (Φ_0, Ψ_0) *is asymptotically stable.*

Proof. Define the perturbation variables

$$\begin{aligned} x_1 &= A, \\ x_2 &= \Phi - \Phi_0, \\ x_3 &= \Psi - \Psi_0, \end{aligned}$$

and expand the equations in a Taylor series about the critical point. In this case,

$$L_0 = \begin{bmatrix} \alpha\psi_c'(\Phi_0) & 0 & 0 \\ 0 & \frac{1}{l_c}\psi_c'(\Phi_0) & -\frac{1}{l_c} \\ 0 & \frac{1}{4B^2 l_c} & -\frac{1}{4B^2 l_c}\Phi_T'(\gamma_0, \Psi_0) \end{bmatrix},$$

$$Q_0(x, y) = \begin{bmatrix} \frac{\alpha}{2}\psi_c''(\Phi_0)(x_1y_2 + x_2y_1) \\ \frac{1}{2l_c}\psi_c''(\Phi_0)(\frac{1}{2}x_1y_1 + x_2y_2) \\ -\frac{1}{8B^2 l_c}\Phi_T''(\gamma_0, \Psi_0)x_3y_3 \end{bmatrix},$$

$$C_0(x,y,z) = \begin{bmatrix} \alpha\psi_c'''(\Phi_0)(\frac{1}{8}x_1y_1z_1 + \frac{1}{6}(x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_1)) \\ \frac{1}{l_c}\psi_c'''(\Phi_0)(\frac{1}{6}x_2y_2z_2 + \frac{1}{12}(x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1)) \\ -\frac{1}{24B^2l_c}\Phi_T'''(\gamma_0,\Psi_0)x_3y_3z_3 \end{bmatrix}.$$

The matrix L_0 has a zero eigenvalue at the peak of the compressor characteristic where $\psi_c'(\Phi_0) = 0$ and the system undergoes a stationary pitchfork bifurcation. The left and right eigenvectors of the zero eigenvalue of L_0 are

$$\begin{aligned} l &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ r^{\mathrm{T}} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\lambda_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{4l_c}\psi_c''(\Phi_0) \\ 0 \end{bmatrix} = 0.$$

Since $\psi_c'(\Phi_0) = 0$ and $\Phi_T'(\gamma_0,\Psi_0) = \frac{\Phi_0}{2\Psi_0}$, δ can be computed from

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l_c} \\ 0 & \frac{1}{4B^2l_c} & -\frac{\Phi_0}{8B^2l_c\Psi_0} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4l_c}\psi_c''(\Phi_0) \\ 0 \end{bmatrix}.$$

Since $l\delta = \delta_1 = 0$, it follows that

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{8}\psi_c''(\Phi_0)\frac{\Phi_0}{\Psi_0} \\ \frac{1}{4}\psi_c''(\Phi_0) \end{bmatrix}.$$

Hence,

$$\lambda_2 = 2\alpha\left[\frac{1}{8}(\psi_c''(\Phi_0))^2\frac{\Phi_0}{\Psi_0} + \frac{1}{8}\psi_c'''(\Phi_0)\right].$$

Using Theorem 3.1, the bifurcated solution is asymptotically stable if $\lambda_2 < 0$, which is equilvalent to (8). ∎

Using the above theorem, it is easy to see that if a system has a subcritical bifurcation for which the bifurcated branch is unstable, it follows that

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) > 0.$$

4.2 Analysis of Control Laws

In this section, we consider four control laws. In each case, the control laws are chosen so that for a positive gain, the control input is always positive. In the sequel we will derive conditions on the gains for the stability of the bifurcated solutions to guarantee that the subcritical pitchfork bifurcation is transformed into a supercritical pitchfork bifurcation. In some cases, the conditions will lead to positive gains while in others the conditions will lead to negative gains. The four control laws under consideration are

$$u = K(\Phi - \Phi_e)^2, \tag{9}$$

$$u = \frac{K}{\sqrt{\Psi}}(\Phi - \Phi_e)^2, \tag{10}$$

$$u = K(\Psi - \Psi_e)^2, \tag{11}$$

$$u = \frac{K}{\sqrt{\Psi}}(\Psi - \Psi_e)^2, \tag{12}$$

where (Φ_e, Ψ_e) is the set point of the system on the compressor characteristic corresponding to the design point and distinct from the critical point (Φ_0, Ψ_0). Note that (10) can be written in terms of a Taylor series about the set point as

$$u = \frac{K\,(\Phi - \Phi_e)^2}{\sqrt{\Psi_e}} - \frac{K\,(\Phi - \Phi_e)^2\,(\Psi - \Psi_e)}{2\,{\Psi_e}^{\frac{3}{2}}} + \frac{3\,K\,(\Phi - \Phi_e)^2\,(\Psi - \Psi_e)^2}{8\,{\Psi_e}^{\frac{5}{2}}} + \mathcal{O}(\Phi - \Phi_e)^3,$$

while (12) can be written as

$$u = \frac{K\,(\Psi - \Psi_e)^2}{\sqrt{\Psi_e}} + \mathcal{O}(\Psi - \Psi_e)^3.$$

The following theorems give conditions for the stability of the closed loop system with the various control laws.

Theorem 4.2 *Consider the control law $u = K(\Phi - \Phi_e)^2$. If*

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) - 2K(\Phi_0 - \Phi_e)\Psi_0^{3/2}\psi_c'''(\Phi_0) < 0, \tag{13}$$

then the bifurcation is supercritical.

Proof. The Taylor series expansion yields

$$L_0 = \begin{bmatrix} \alpha\psi_c'(\Phi_0) & 0 & 0 \\ 0 & \frac{1}{l_c}\psi_c'(\Phi_0) & -\frac{1}{l_c} \\ 0 & \frac{1}{4B^2 l_c}(1 - 2K(\Phi_0 - \Phi_e)\sqrt{\Psi_0}) & -\frac{\Phi_0}{8B^2 l_c \Psi_0} \end{bmatrix}.$$

As for the open loop system, $\lambda_1 = 0$ and

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l_c} \\ 0 & \frac{1}{4B^2 l_c}(1 - 2K(\Phi_0 - \Phi_e)\sqrt{\Psi_0}) & -\frac{\Phi_0}{8B^2 l_c \Psi_0} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4l_c}\psi_c''(\Phi_0) \\ 0 \end{bmatrix}.$$

Since $l\delta = \delta_1 = 0$, it follows that

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\Phi_0 \psi_c''(\Phi_0))}{8\Psi_0(1-2K(\Phi_0-\Phi_e)\sqrt{\Psi_0})} \\ \frac{1}{4}\psi_c''(\Phi_0) \end{bmatrix}.$$

Finally,

$$\lambda_2 = 2\alpha \left[\frac{\Phi_0(\psi_c''(\Phi_0))^2}{8\Psi_0(1 - 2K(\Phi_0 - \Phi_e)\sqrt{\Psi_0})} + \frac{1}{8}\psi_c'''(\Phi_0) \right],$$

so that the bifurcated solution is stable and the bifurcation is supercritical if (13) is satisfied. ■

Theorem 4.3 *Consider the control law* $u = \frac{K}{\sqrt{\Psi}}(\Phi - \Phi_e)^2$. *If*

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) - K\left[(\psi_c''(\Phi_0))^2(\Phi_0 - \Phi_e)^2 + 2\Psi_0(\Phi_0 - \Phi_e)\psi_c'''(\Phi_0)\right] < 0, \tag{14}$$

then the bifurcation is supercritical.

Theorem 4.4 *Consider the control law* $u = K(\Psi - \Psi_e)^2$. *If*

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) + 4K(\Psi_0 - \Psi_e)\Psi_0^{3/2}(\psi_c''(\Phi_0))^2 < 0, \tag{15}$$

then the bifurcation is supercritical.

Theorem 4.5 *Consider the control law* $u = \frac{K}{\sqrt{\psi}}(\Psi - \Psi_e)^2$. *If*

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) + K(\psi_c''(\Phi_0))^2\left[4\Psi_0(\Psi_0 - \Psi_e) - (\Psi_0 - \Psi_e)^2\right] < 0, \tag{16}$$

then the bifurcation is supercritical.

The proofs of Theorems 4.3-4.5 are similar to that of Theorem 4.2 and are omitted.

4.3 Alternative analysis of control laws

The conditions in Theorems 4.2-4.5 can be found using an alternative technique. First, introduce the variable $J = A^2$ as the square of the rotating stall first Fourier magnitude. The equations can then be written as

$$
\begin{aligned}
\frac{dJ}{d\xi} &= 2\alpha J(\psi_c'(\Phi) + \frac{J}{8}\psi_c'''(\Phi)),\\
\frac{d\Phi}{d\xi} &= \frac{1}{l_c}(\psi_c(\Phi) - \Psi + \frac{J}{4}\psi_c''(\Phi)),\\
\frac{d\Psi}{d\xi} &= \frac{1}{4B^2 l_c}(\Phi - \Phi_T(\gamma, \Psi) - u\sqrt{\Psi}).
\end{aligned}
$$

The equilibrium conditions are

$$
\begin{aligned}
\psi_c'(\Phi) &= -\frac{J}{8}\psi_c'''(\Phi),\\
\Psi &= \psi_c(\Phi) + \frac{J}{4}\psi_c''(\Phi),\\
\Phi &= (\gamma + u)\sqrt{\Psi}.
\end{aligned}
$$

The pitchfork bifurcation of the system written in the A coordinate is transformed to a transcritical bifurcation when the J coordinate is used [6]. In addition, the former is subcritical (supercritical) when the latter is subcritical (supercritical). It is noted from the geometry that the transcritical bifurcation becomes supercritical when $d\gamma/dJ < 0$. Hence, the stability conditions from the previous section can be rederived by finding the condition for which $d\gamma/dJ < 0$.

The derivatives of the first and second equations at the equilibrium yield

$$
\begin{aligned}
\psi_c''(\Phi_0)\frac{d\Phi}{dJ} &= -\frac{1}{8}\psi_c'''(\Phi_0),\\
\frac{d\Phi}{dJ} &= -\frac{\frac{1}{8}\psi_c'''(\Phi_0)}{\psi_c''(\Phi_0)},
\end{aligned}
$$

and

$$
\frac{d\Psi}{dJ} = \frac{1}{4}\psi_c''(\Phi_0).
$$

Now, consider the controller (9). The new equilibrium condition is

$$
\Phi = (\gamma + K(\Phi - \Phi_e)^2)\sqrt{\Psi}.
$$

The derivative of the new equilibrium condition is

$$\frac{d\Phi}{dJ} = \sqrt{\Psi_0}\left(\frac{d\gamma}{dJ} + 2K(\Phi_0 - \Phi_e)\frac{d\Phi}{dJ}\right) + \frac{\gamma_0 + K(\Phi_0 - \Phi_e)^2}{2\sqrt{\Psi_0}}\frac{d\Psi}{dJ}.$$

It follows that

$$(1 - 2K\sqrt{\Psi_0}(\Phi_0 - \Phi_e))\frac{d\Phi}{dJ} = \sqrt{\Psi_0}\frac{d\gamma}{dJ} + \frac{\Phi_0}{2\Psi_0}\frac{d\Psi}{dJ}.$$

Hence if $d\gamma/dJ < 0$ then

$$(1 - 2K\sqrt{\Psi_0}(\Phi_0 - \Phi_e))\frac{\frac{1}{8}\psi_c'''(\Phi_0)}{\psi_c''(\Phi_0)} + \frac{\Phi_0}{8\Psi_0}\psi_c''(\Phi_0) > 0.$$

Noting that $\psi_c''(\Phi_0) < 0$, it follows that

$$(1 - 2K\sqrt{\Psi_0}(\Phi_0 - \Phi_e))\Psi_0\psi_c'''(\Phi_0) + \Phi_0(\psi_c''(\Phi_0))^2 < 0,$$

which is equivalent to (13). Similar results can be obtained for conditions (14)-(16).

4.4 Critical Gains

As mentioned before, each of the control laws must be postive, that is, $u \geq 0$ for all values of the feedback variables. In each case, the control laws are quadratic functions of the feedback variables so that the control is positive only if the corresponding gain K is positive. The conditions (13)-(16) for each of the control laws are inequalities that depend on the gains K. Hence, the inequalities can be used to determine whether a positive K satisfies the condition.

Theorem 4.6 *The control laws (9) and (10) satisfy the conditions (13) and (14), respectively, with some $K > 0$.*

Proof. First note that when the open loop system has a subcritical bifurcation,

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) > 0.$$

Now consider (13). Because of the shape of the compressor characteristic, $\psi_c'''(\Phi_0) < 0$. Further, since the set point Φ_e is greater than the peak value Φ_0, it follows that $\Phi_0 - \Phi_e < 0$. Hence, (13) can be rewritten as

$$0 < \frac{(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0)}{2(\Phi_0 - \Phi_e)\Psi_0^{3/2}\psi_c'''(\Phi_0)} < K.$$

Next consider (14). As before, $\psi_c'''(\Phi_0) < 0$ and $\Phi_0 - \Phi_e < 0$. Hence, (14) can be rewritten as

$$0 < \frac{(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0)}{(\psi_c''(\Phi_0))^2(\Phi_0 - \Phi_e)^2 + 2\Psi_0(\Phi_0 - \Phi_e)\psi_c'''(\Phi_0)} < K.$$

■

Theorem 4.7 *The control laws (11) and (12) satisfy the conditions (15) and (16), respectively, with some $K < 0$.*

Proof. Consider (15). Since the set point Ψ_e is less than the peak value Ψ_0, it follows that $\Psi_0 - \Psi_e > 0$. Hence, (15) can be rewritten as

$$0 > -\frac{(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0)}{4(\Psi_0 - \Psi_e)\Psi_0^{3/2}(\psi_c''(\Phi_0))^2} > K.$$

Next note that (16) can be rewritten as

$$(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0) + K(\psi_c''(\Phi_0))^2(\Psi_0 - \Psi_e)(3\Psi_0 + \Psi_e) < 0.$$

so that

$$0 > -\frac{(\psi_c''(\Phi_0))^2\Phi_0 + \Psi_0\psi_c'''(\Phi_0)}{(\psi_c''(\Phi_0))^2(\Psi_0 - \Psi_e)(3\Psi_0 + \Psi_e))} > K.$$

■

From these two theorems we see that control laws (9) and (10) will stabilize the rotating stall branch for $u > 0$ with a positive gain K, while the control laws (11) and (12) only stabilize the rotating stall branch with a negative gain K so that $u < 0$. Because of physical constraints on the bleed valve actuation, only (9) and (10) represent useful feedback control laws. Next, we consider the range of gains for the control laws (9) and (10) that stabilize the linearization of the system at the peak of the characteristic.

Theorem 4.8 *The control law (9) stabilizes the linearization of the bifurcated solution of (1)-(3) over a semi-infinite interval.*

Proof. From Theorem 4.2 we know that control law (9) stabilizes the bifurcated solution for some positive interval. Consider the linearization at the peak,

$$L_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l_c} \\ 0 & \frac{1}{4B^2 l_c}(1 - 2K(\Phi_0 - \Phi_e)\sqrt{\Psi_0}) & -\frac{\Phi_0}{8B^2 l_c \Psi_0} \end{bmatrix}.$$

The nonzero eigenvalues of the linearization must remain in the open left half of the complex plane for the linearized system and hence the nonlinear system to be stable. The characteristic equation for the partition of the linearization with nonzero eigenvalues is

$$\lambda^2 + \frac{\Phi_0}{8B^2 l_c \Psi_0}\lambda + \frac{1}{4B^2 l_c^2}(1 - 2K(\Phi_0 - \Phi_e)\sqrt{\Psi_0}) = 0.$$

Hence, for stability,

$$1 - 2K(\Phi_0 - \Phi_e)\sqrt{\Psi_0} > 0,$$

which requires

$$\frac{1}{2(\Phi_0 - \Phi_e)\sqrt{\Psi_0}} < K,$$

so that the bifurcated solution is stabilized for K over a semi-infinite interval. ∎

Theorem 4.9 *The control law (10) stabilizes the linearization of the bifurcated solution of (1)-(3) over a finite interval with*

$$K < \frac{\Phi_0}{(\Phi_0 - \Phi_e)^2}. \tag{17}$$

Proof. From Theorem 4.3 we know that control law (10) stabilizes the bifurcated solution for some positive interval. Consider the linearization at the peak,

$$L_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l_c} \\ 0 & \frac{1}{4B^2 l_c}(1 - 2K(\Phi_0 - \Phi_e)) & -\frac{(\Phi_0 - K(\Phi_0 - \Phi_e)^2)}{8B^2 l_c \Psi_0} \end{bmatrix}.$$

The nonzero eigenvalues of the linearization must remain in the open left half of the complex plane for the linearized system and hence the nonlinear system to be stable. The characteristic equation for the partition of the linearization with nonzero eigenvalues is

$$\lambda^2 + \frac{(\Phi_0 - K(\Phi_0 - \Phi_e)^2)}{8B^2 l_c \Psi_0}\lambda + \frac{1}{4B^2 l_c^2}(1 - 2K(\Phi_0 - \Phi_e)) = 0.$$

Hence, for stability,

$$1 - 2K(\Phi_0 - \Phi_e) > 0,$$

and

$$\Phi_0 - K(\Phi_0 - \Phi_e)^2 > 0,$$

which require

$$\frac{1}{2(\Phi_0 - \Phi_e)} < K,$$

and (17). Hence, the bifurcated solution is stabilized for K over a finite interval. ∎

5 Numerical Example

In this section, we present a numerical example to demonstrate the two control laws which were shown in the previous section to stabilize the rotating stall branch, make the bifurcation supercritical, and eliminate the hysteresis, namely, (9) and (10). Specifically, we show that the hysteresis is eliminated using the critical values of the gains for which expressions were derived in the previous section, and that for larger values of the gain the bifurcation is softened further.

5.1 Bifurcation Analysis

Consider the compressor having characteristic

$$\psi_c(\Phi) = 0.149 - 1.017\Phi + 13.510\Phi^2 - 16.078\Phi^3.$$

The peak value of this characteristic is $(\Phi_0, \Psi_0) = (0.52, 1.01)$. The bifurcation diagram for the open loop system for the rotating stall first Fourier magnitude A as a function of the throttle parameter γ appears in Figure 2 for $B = 0.1$, $\alpha = 0.3$, and $l_c = 6.66$. The stable portions of the rotating stall and axisymmetric branches are plotted conventionally with solid curves while the unstable portions are plotted with dashed curves. Note the subcritical pitchfork bifurcation and the hysteresis region, where there is a range of the throttle parameter γ such that more than one stable equilibria exist. We will use the controllers of the previous section to eliminate the hysteresis.

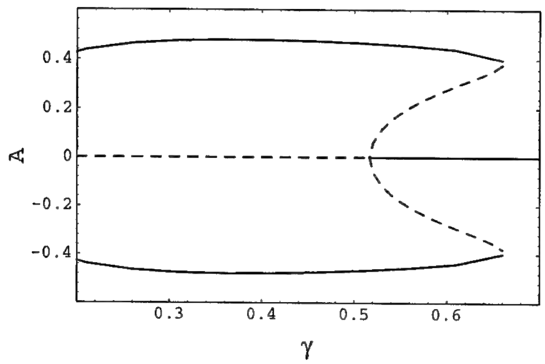

Figure 2: Bifurcation diagram for the open loop system

5.1.1 Controller 1

First, consider the controller $u = K(\Phi - \Phi_e)^2$. From the previous section, the critical value of the gain K depends upon the value of the set point parameter Φ_e. In this case, the gain must satisfy

$$K > \frac{179.7}{196.5\Phi_e - 102.1},$$

to eliminate the hysteresis. A plot of the critical value of the gain versus the set point is shown in Figure 3. For the rotating stall branch to be stabilized, the gain must lie above the curve. Note that as the set point approaches the value corresponding to the peak of the compressor characteristic, the critical value of the gain increases. Hence, a larger gain is required for a set point closer to the peak and operation at the peak would require an infinite gain since the quadratic term would then play no role in the stability of the bifurcated solution.

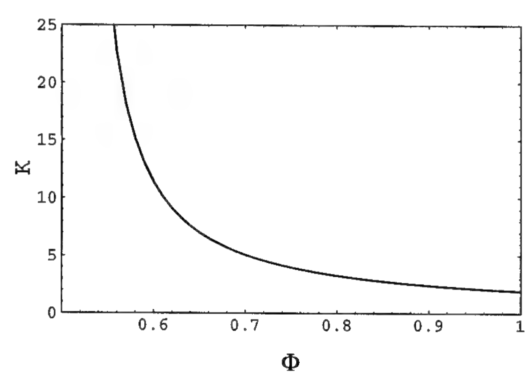

Figure 3: Critical value of gain versus set point Φ_e for (9)

Next, we choose the set point to correspond to the throttle gain $\gamma = 0.6$. The corresponding point along the compressor characteristic is $(\Phi_e, \Psi_e) = (0.59, 0.96)$. The critical value of the gain K that makes the bifurcation is supercritical is 13.66. Bifurcation plots of the closed loop system for this value and for $K = 25$ are shown in Figure 4. Note that the hysteresis is eliminated and that the pitchfork bifurcation softens as the gain is increased. Note also that the value of the throttle parameter at which the bifurcation occurs has decreased from the open loop value.

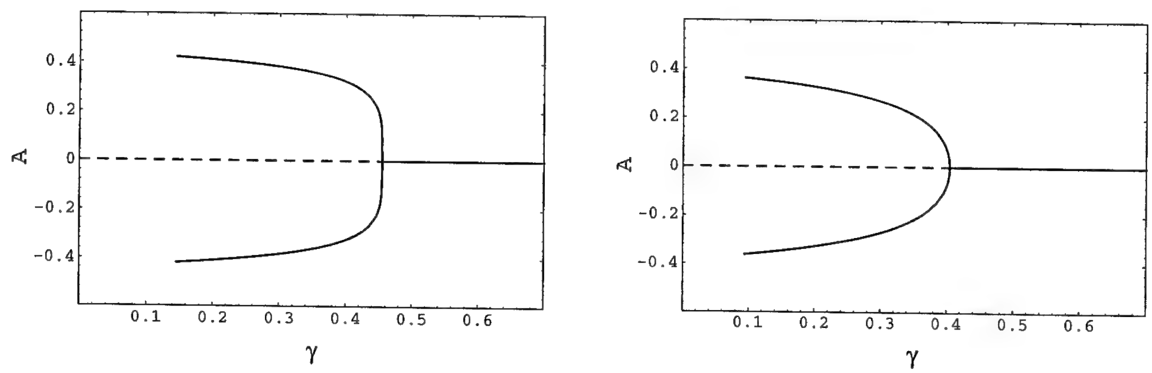

Figure 4: Bifurcation diagrams for controller (9), $K = 13.66, 25$

5.1.2 Controller 2

Next consider the controller $u = \frac{K}{\sqrt{\Psi}}(\Phi - \Phi_e)^2$. In this case, the hysteresis is eliminated and the rotating stall branch stabilized if

$$\frac{179.7}{533.8\Phi_e^2 - 359.4\Phi_e + 42.6} < K < \frac{0.52}{\Phi_e^2 - 1.04\Phi_e + 0.27}.$$

A plot of the critical value of the gain as a function of the set point is shown in Figure 5. Since this controller stabilizes the bifurcated solution over a finite interval, the gain must lie between these two curves. The set point is chosen as before as $(\Phi_e, \Psi_e) = (0.59, 0.96)$ corresponding to $\gamma = 0.6$. Bifurcation diagrams for the closed loop system with the critical value of the gain, $K = 11.62$, and $K = 25$ are shown in Figure 6. In this case, $K < 115.9$ is required for stability of the bifurcated solution. We see from the bifurcation diagram in Figure 6 that this controller creates a new unstable rotating stall equilibrium point. The origin of this new equilibrium point will be explained in a graphical interpretation of the equilibrium equations in the state space in the sequel.

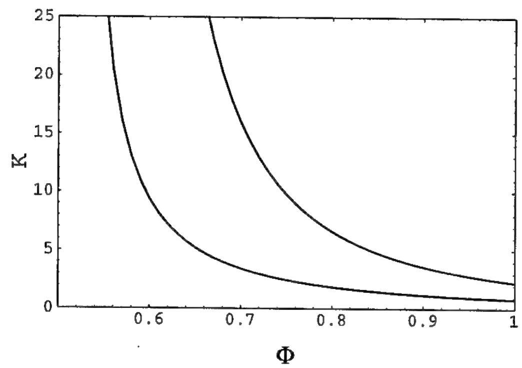

Figure 5: Critical value of gain versus set point Φ_e for (10)

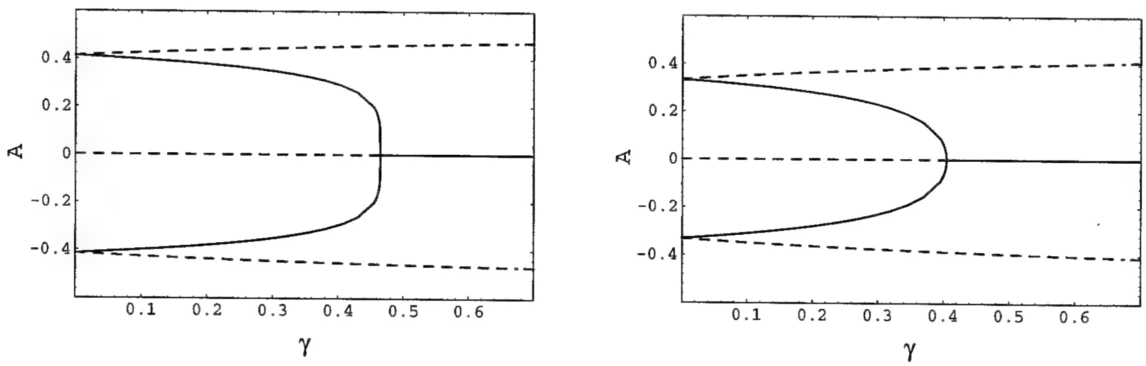

Figure 6: Bifurcation diagrams for controller (10), $K = 11.62, 25$

5.2 Geometric Interpretation

The axisymmetric and rotating stall branches of the equilibria are shown in the three dimensional state space (A, Φ, Ψ) in Figure 7. On the left is a plot of the equilibrium branches, with the axisymmetric branch in the $A = 0$ plane shown as the darker line. On the right the throttle nonlinearity

$$\Phi^2 = \gamma^2 \Psi,$$

for $\gamma = 0.6$ is shown as a curved surface intersecting the equilibrium branches. Note that the throttle surface intersects both branches so that hysteresis is present for this value of the throttle parameter. As the throttle parameter γ is varied, the throttle surface moves up and down by increasing and decreasing its steepness. The pitchfork bifurcation occurs as the surface passes

through the peak.

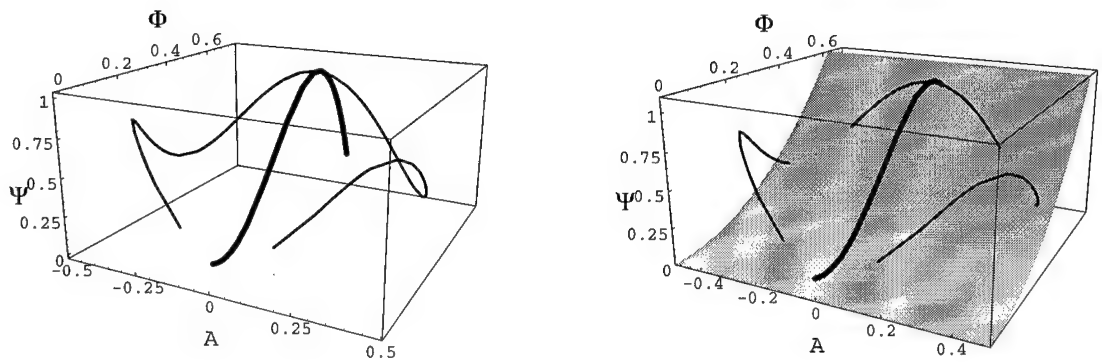

Figure 7: Equilibrium Branches in State Space

The bleed valve control laws used in this paper can be interpreted graphically as modifying the throttle surface, hence modifying the behavior of the closed loop system. We can characterize the modified throttle surface for control law (9) in terms of the gain K. In this case, the modified throttle surface is

$$\Phi^2 = (\gamma + K(\Phi - \Phi_e)^2)^2 \Psi.$$

This new surface is shown with the equilibrium branches in Figure 8. Note that the modified throttle surface no longer intersects both branches, but only the axisymmetric branch. Now, as the throttle surface moves up and passes through the characteristic peak, it only intersects the rotating stall branch past the peak, hence the supercritical bifurcation.

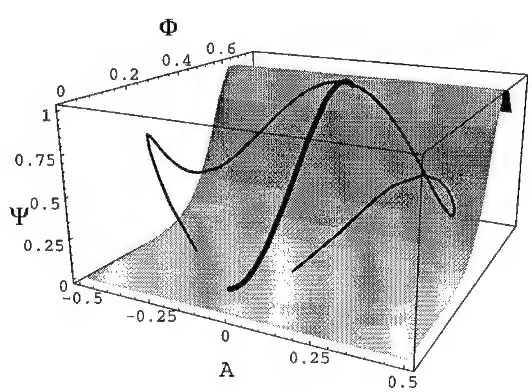

Figure 8: Equilibrium Branches in State Space with Control Law (9)

Finally consider the control law (10). The modified throttle surface in this case is

$$(\Phi - K(\Phi - \Phi_e)^2)^2 = \gamma^2 \Psi.$$

The new surface is shown as before in Figure 9. Two different views are shown for clarity, the one on the left with only a portion of the modified throttle surface and the one on the right with the entire surface in the range of the plot. Note that the modified throttle surface now goes through a minimum and then intersects the rotating stall branch. Hence, the new equilibrium point is created. The variation in the modified throttle surface is also worth mentioning to explain the nature of the bifurcation diagram in Figure 6. As γ decreases, both sides of the throttle surface increase in steepness so that for very small γ, the throttle surface appears as a very narrow valley intersecting the rotating stall branch at two close points. This explains why the equilibrium values of the rotating stall equilibria in Figure 6 converge as γ approaches zero.

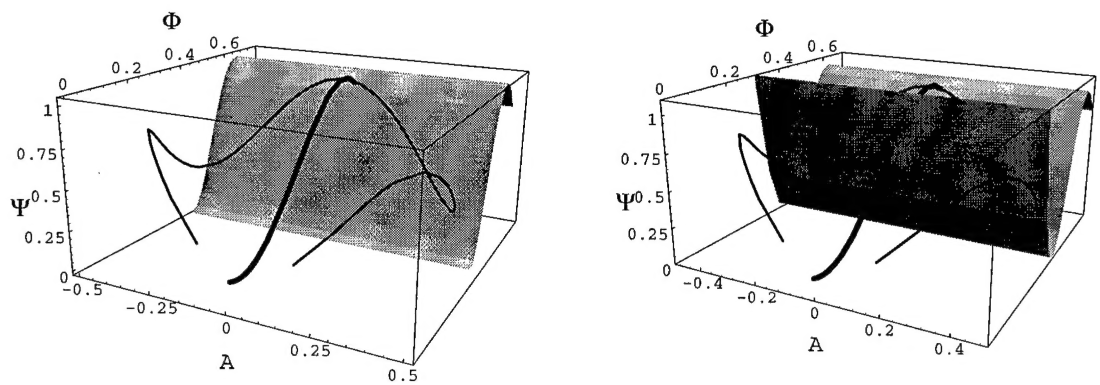

Figure 9: Equilibrium Branches in State Space with Control Law (10)

6 Discussion

In this paper we have extended bifurcation stabilization results for a system whose critical mode is uncontrollable and unobservable to the case where the control does not vanish at the critical point. Control about a set point or operating point is considered, where the set point and the critical point are distinct. Such a control makes physical sense with systems such as compressors where operation away from the critical point is desired but the stability of the bifurcated solution is important because subcritical bifurcations can lead to hysteresis and undesirable behavior. The main result showed that each of the terms of the control law written in a Taylor series about the set point affected the stability of the bifurcated solution. Hence, control laws could be chosen to have a structure to meet other requirements. In this case, control laws for the compressor model were chosen to give only positive values of the control since bleed valve actuation is considered. It was shown that two control laws using only annulus-averaged quantities as feedback variables and bleed valves for actuation eliminate the hysteresis associated with rotating stall and stabilize the rotating stall equilibrium branch of the Moore-Greitzer three state model. A numerical example was used to show that as the gain increases beyond the critical value, the bifurcation is further softened. Furthermore, geometric interpretations of the control laws were given in the three dimensional state space.

The controllers shown here have advantages over other control laws that appear in the literature. The control law $u = KA^2$ introduced in [6] achieves the same result, but with several disadvantages. The use of the first Fourier magnitude A as a feedback variable increases the sensing and signal processing requirements of the control system, and hence increases the complexity of the seemingly simple feedback law. In addition, the authors in [6] consider a general nonlinear function of all of the feedback variables and conclude that only the square of the rotating stall magnitude affects the bifurcation. However, only the error states from the stall point are considered, rather than the error states using the set point as used here. In fact, if the set point in the present work is set equal to the stall point, then the results match those of [6]. Finally, use of the control law in [6] can have no effect on deep surge, as the control law is identically zero in the invariant $\Phi - \Psi$ plane in which a surge limit cycle exists. Furthermore, such a control law may have detrimental effects on classic surge, or coupled rotating stall and surge, as described in [2]. The effect of the present control laws on surge will be the subject of future study.

The use of only annulus-averaged quantities in a control law for rotating stall was also described in [5], in which backstepping was used to find a control law that globally stabilized a set point. This work showed that hysteresis could be eliminated without feeding back the rotating stall magnitude. However, the derivation of the backstepping control law is complicated and results in a control law whose control action is not guaranteed to be positive for all values of the feedback variables and whose control input at the set point does not equal zero. In addition, although the avoidance of cancellation of nonlinearities is lauded as a benefit to the backstepping control law design technique, the formulation of the problem results in cancellation of the throttle nonlinearity by assuming that the throttle parameter γ is the control law rather than augmenting γ with a control input term.

7 Acknowledgements

The authors would like to gratefully thank Wei Kang for many helpful discussions on bifurcation stabilization.

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